SIACG 2002 – 1st Ibero–American Symposium on Computer Graphics
1–5 July 2002, Guimarães – Portugal

Free–Form Solid Modelling Based on Extended Simplicial Chains Using Triangular Bézier Patches

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Abstract

We present a mathematical model for geometric modelling based on the concept of extended simplicial chain (ESC) defined in previous works. With this model, a solid is defined by means of an algebraic sum of non–disjoint extended cells, applying the divide and conquer concept. This allows us to obtain the traditional Boolean operations in geometric modelling through the operations defined for ESC's. The model enables us to represent free–form solids whose boundaries are free–form surfaces represented by a set of low degree triangular Bézier patches and operate with them. In fact, this model allows us to solve basic problems in solid modelling, like the point–in–solid test. In this case we make use of the generality of the definition of ESC to particularize it to the use of triangular Bézier patches in 3D.

Keywords

Free–form solid modelling, extended simplicial chains, triangular Bézier patches, formal methods in computer graphics, mathematical models.

1. INTRODUCTION

The use of mathematical models for geometric modelling provides important advantages as mentioned in [Duce91]: homogeneous study of the representation system properties and ease of implementation.

This work presents a mathematical model to represent free–form solids (i.e., solids whose boundary is defined by elements of free nature called *free–form elements:* curves in 2D, surfaces in 3D, etc.). The main problem to achieve this is to deal with the non–planar zones in a simple and homogeneous way.

In [Ruiz99, Ruiz01a] a mathematical model for geometric modelling based on *extended simplicial chains* (ESC's) is presented. In those works, the authors apply the model to solids in 2D (considered as bounded either by cubic Bézier curves or conic arcs), and to solids in 3D, considering them as bounded by a free–form surface expressed as a set of algebraic patches. The model makes use of what it calls *free–form cells* (ffc's) to simplify the representation an to avoid particular cases.

The present work makes use of the ESC concept too by defining a new kind of ffc using triangular Bézier patches (TBP's), and adapts the definitions and theorems to use the ESC's with the new kind of ffc as a mathematical model to represent free–form solids. With this model it is easy to develop fundamental algorithms in solid modelling, like point–in–solid test (see [García01]).

Next we are going to review the previous work: other methods proposed to formalize and manage free–form solids, the concept of ffc, and the ESC's. After that, we will particularize the ffc concept to use the TBP's. Finally, we will show the way to represent free–form solids and Boolean operations with ESC's.

2. PREVIOUS WORK

The formalization and management of free–form solids representations is not an easy task. In fact, there are many works that present different formal methods to achieve this. One of them is the *R–function* model from Shapiro [Shapiro94]; this model can be used for interactive modelling or animation with *blobs* [Blinn82, Wyvil95], but it can be very difficult to define implicit real functions for free–form solids defined by means of free–form surfaces constructed as piecewise surface patches.

Other interesting works are [Kumar95] and [Keyser97]. The construction, evaluation and visualization of CSG complex models using primitives described by parametric patches (NURBS and *trimmed Bézier patches*) is studied in these papers. The first reference treats the visualization of the solids, having to solve the point–in–solid test in order to calculate the solid surface; this is made by means of solving ray–patch intersections, with precission and robustness problems. The second reference uses exact arithmetic to reduce the problems from the first one, but that implies an important reduction of efficiency.

The representations based on multiresolution subdivision surfaces have been studied too, and in [Biermann01] a method for computing approximate results of Boolean operations for free–form solids bounded by this kind of patches is described. This algorithm works with the control point meshes of the boundaries of the solids to calculate an approximation of the intersection curve for the resulting solid.

The TBP's have been used previously in other works, like the one by Kolb, Pottmann and Seidel [Kolb95], who developed an algorithm with two variants based on 7–degree and 4–degree TBP's to reconstruct the surface of a free–form solid starting from a polyhedral approximation.

The ESC's model try to solve the free–form solid representation by using a *divide and conquer* approach, considering the solids as composed by a set of elements called *extended cells*, and developing uniform and robust methods to carry out the basic operations in solid modelling. Next we are going to review the basis of this model.

2.1 General Free–Form Cell

The concept of free–form cell is the free–form equivalent to the planar simplex (triangle in 2D, tetrahedron in 3D, etc.) used in [Feito98]. The formal definition of ffc does not depend on the dimension nor the surface type used; see [Ruiz99, Ruiz01a]. Here we will use TBP's to particularize the ffc.

We can intuitively define a free–form cell [Ruiz01a] as the closed set of points (in the dimension we are working) limited by one free–form element and several *planar elements* (line segments in 2D, polygons/planes in 3D, etc.). The number and type of the planar elements that bound a ffc depend on the particular free–form element used to define it, and they are common to other simplices or ffc's used to define the solid.

Definition 1. d–Dimensional *free–form cell* (ffc) is the set of points in $\mathbb{R}^d$ obtained as the intersection of the half–spaces defined by a free–form element of dimension *d* and one or more planar elements of dimension *d–1* that verifies:

1. It is a closed and connected set in $\mathbb{R}^d$.
2. Every point lies on the same connected component (sign–invariant) with regard to the implicit equation of the free–form element, except the ones of the boundary.
3. The points common to a number of bound elements greater than or equal to the dimension of the ffc belong to the free–form element too. ♠

The first condition avoids configurations as the one shown in figure 1.a to be considered as ffc's, because they can be divided into simpler ffc's, like the ones in figure 1.b.

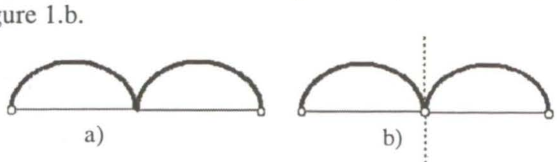

Figure 1. First condition for ffc's

The set of points may be closed and connected, but sometimes its points are not in the same connected component (see an example in figure 2.a). The second condition explained for ffc's excludes these situations, that can be solved by means of the division (more or less easily) of the set of points, like shown in figure 2.b.

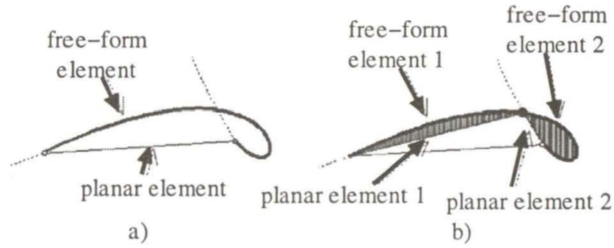

Figure 2. Second condition for ffc's

The third condition gives us a method to select the vertices of the ffc (see figure 3.a), establishing that the vertices of a ffc must belong to the free–form element. It is also applied to discard candidates to ffc that verify the first two conditions (see figure 3.a), and eliminates configurations that are too complex and can be simplified (see figure 3.b).

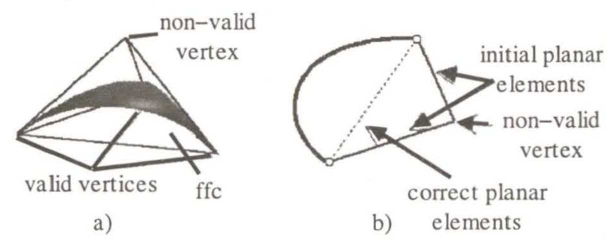

Figure 3. Third condition for ffc's

Any ffc has an associated sign (+1 or –1) depending on the orientation of the ffc with regard to the solid. If the associated sign is +1, then the ffc is added to the solid; if the sign is –1, then the ffc is substracted to the solid. The sign is used in the definition of both solids and operations with them.

2.2 Extended Simplicial Chains

The concept of extended simplicial chain (ESC) is an extension of the simplicial chain introduced in [Feito98], using ffc's. Next we will remember the formal definition of ESC.

Definition 2. A d–dimensional *extended simplicial chain* δ is defined as the expression:

$$\delta = \sum_i \alpha_i \cdot E_i$$

where the E_i are *extended cells*, each of them multiplied by an integer α_i (its coefficient). An extended cell may be either a simplex or a ffc. ♠

For our purposes, we take coefficients of value –1, 0 or 1 (a value of 1 indicates presence of the solid). The sign of a simplex is calculated using the signed area (in 2D), or the signed volume (in 3D), see [Feito98]. When a simplex has a vertex in the origin, it is called an *original simplex.*

To identify the free–form solid represented by an ESC, we make use of a function associated with the chain, defined as follows:

Definition 3. Let δ be an ESC. The *associated function of the chain* δ, noted f_δ is:

$$f_\delta : \mathbb{R}^d \rightarrow \mathbb{Z}$$
$$f_\delta(Q) = \sum_{Q \in E_i} \alpha_i$$ ♠

In this expression, **Q** is a point in $\mathbb{R}^d$, and then f_δ is equal to the sum of the coefficients associated to the extended cells that contains the point **Q**. Based on this function, we can define the solid associated with an ESC as:

Definition 4. The d–dimensional *free–form solid associated with the ESC* δ, noted FF_δ is:

$$FF_\delta = \{Q \in \mathbb{R}^d / f_\delta(Q) \neq 0\}$$ ♠

Once we have reviewed the concepts of ESC and its associated solid, we define the operations over ESC's in the following way:

Definition 5. Let δ and δ' be two ESC's as described in definition 2, and let λ be a scalar value. The sum of ESC's and the product of an ESC by a scalar are defined as:

$$\delta = \sum_{i=1}^{m} \alpha_i \cdot E_i; \quad \delta' = \sum_{i=1}^{m'} \alpha'_i \cdot E'_i;$$
$$\delta + \delta' = \sum_{i=1}^{m} \alpha_i \cdot E_i + \sum_{i=1}^{m'} \alpha'_i \cdot E'_i;$$ ♠
$$\lambda \cdot \delta = \sum_{i=1}^{m} (\lambda \cdot \alpha_i) \cdot E_i;$$

In order to regularize the operations so that problems in zones common to extended cells with coefficients of opposite sign do not arise, the associated regular function of an ESC is defined:

Definition 6. Let δ be an ESC, f_δ its associated function, and FF_δ its associated free–form solid. The *associated regular function* of the chain δ, noted f_δ^* is:

$$f_\delta^* : \mathbb{R}^d \rightarrow \mathbb{Z}$$
$$f_\delta^*(\mathbf{Q}) = 0, \quad \text{if } \mathbf{Q} \notin cl(int(FF_\delta))$$
$$f_\delta^*(\mathbf{Q}) = p, \quad \text{if } p > 0$$
$$f_\delta^*(\mathbf{Q}) = n, \quad \text{otherwise}$$

where $p = \max \{f_\delta(\mathbf{Q'}) / \mathbf{Q'} \in \varepsilon(\mathbf{Q})\}$, $n = \min \{f_\delta(\mathbf{Q'}) / \mathbf{Q'} \in \varepsilon(\mathbf{Q})\}$, *cl* is the closure operation, *int* represents the interior points, and ε is a neighbourhood small enough. ♠

The regularized operations with ESC's are defined using the associated regular function concept in this way:

Definition 7. Let δ and δ' be two ESC's, and λ a scalar value. The associated regular functions for the sum of δ and δ', and for the product of δ by λ are respectively:

$$f_\delta +^* f_{\delta'} = f^*_{\delta+\delta'}$$
$$\lambda \cdot^* f_\delta = f^*_{\lambda \cdot \delta'}$$ ♠

The set of regular ESC's with these regular operations has a vectorial space structure. See [Feito98, Ruiz01a] for more details. From this point, we are going to use **always regularized operations**, even when we do not specify it clearly.

Although apparently complicated, the regularization does not make more difficult the implementation. Moreover, the implementation of the operations adapts naturally to the regularization.

3. FREE–FORM CELL DEFINED BY TRIANGULAR BÉZIER PATCHES

It is necessary to use a surface in order to define a ffc in 3D. As it is explained in [Ruiz99, Ruiz01a, Ruiz01b], the concept of ffc does not depend on the choice of the type of surface to be used, but for each choice it is necessary to study the conditions to hold definition 1 and to carry out methods to determine the relative position of a point with regard to the ffc defined by the surface chosen. In this case, we are going to use cubic triangular Bézier patches (TBP's) [Farin86, Böhm84, Farin93, Vlachos01, Seidel92].

3.1 Free–Form Surfaces Defined by Means of TBP's

TBP's were the first extension of the Bézier curves to surfaces, as a more *natural* generalization than tensor product patches are. A n–degree TBP is defined by a triangular control point net with `½*(n+1)*(n+2)` points. Each control point is named b_{ijk}, with $i \geq 0$, $j \geq 0$, $k \geq 0$ and $i+j+k=n$.

The parametric domain is triangular, having three parameters: *u*, *v* and *w*, whose values go from 0.0 to 1.0; the sum *u+v+w* equals 1.0 always. Figure 4 shows the control point net of a 3–degree patch, and its parametric domain.

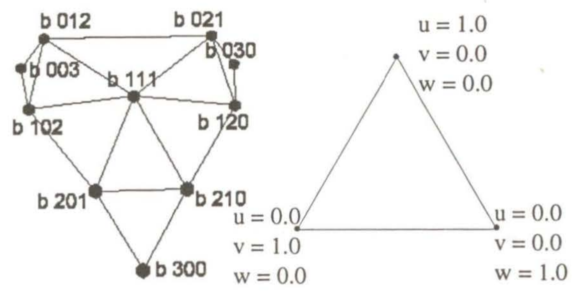

Figure 4. Control point net and parametric domain

To calculate the surface, we use the Bernstein polynomials; for this kind of patch, these polynomials are given by the expression:

$$B^n_{i,j,k}(u,v,w) = \binom{n}{i,j,k} \cdot u^i \cdot v^j \cdot w^k = \frac{n!}{i! \cdot j! \cdot k!} \cdot u^i \cdot v^j \cdot w^k$$

n being the degree of the polynomial.

Calling λ the 3–tuple *(i,j,k)* and τ the 3–tuple *(u,v,w)*, we obtain the expression for a n–degree TBP as:

$$b^n(\tau) = \sum_{|\lambda|=n} \boldsymbol{b}_\lambda \cdot B^n_\lambda(\tau)$$

where $|\lambda|=n$ stands for all the 3–tuples *(i,j,k)* that verify that *i+j+k=n*. Figure 5 shows a 3–degree TBP with its associated control point net. The triangle b_{00n}, b_{0n0}, b_{n00} is called the *base triangle* of the patch.

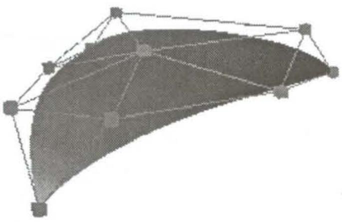

Figure 5. 3-degree triangular Bézier patch

Some of the properties of the TBP's that will be useful later are the following:

- Every TBP is included in the convex hull of its control points.
- Every TBP interpolates b_{n00}, b_{0n0} and b_{00n} (i.e. the vertices of the base triangle).
- The edges of the patch are Bézier curves whose control points are the ones in the edges of the control point mesh.

We use the method described in [Vlachos01] to construct the surface that defines the free–form solid boundary starting from an initial triangulation. The algorithm, thought to be implemented in hardware (in fact, it is the core of the TRUFORM technology used in several graphics chips from ATI), is very simple and efficient, able to process each triangle independently and obtain a patch for each of them. It uses a quadratic interpolation to obtain the normal vector at each point of the patch, and calculates the control point mesh for 3–degree TBP's from the vertexes and vertex normals of the initial triangulation. The joint between the patches obtained has C^0 continuity (C^1 at the corners of the patches), and the resulting surface is visually smooth. The reason for using this algorithm is its simplicity and good adaptability to the fields of application of ESC's (efficient algorithms on free–form solids). Nevertheless, the definition of the ffc, and the free–form solid representation that we will describe are general enough to be applied to any representation using TBP's of any degree.

3.2 Free–Form Cell Defined by TBP's

Once we have obtained the TBP's that compose the free–form surface of the solid, we are going to describe the ffc defined by TBP's. First of all, we will define an element necessary for the ffc definition.

Definition 8. Let t_1 and t_2 be two triangles from an initial triangulation of a free–form solid that share a common edge *e*. The *plane associated to t_1 and t_2*, noted $\pi(t_1, t_2)$, is the plane that contains the edge *e* and is parallel to the average vector of the normal vectors from t_1 and t_2. ♠

Our definition of ffc uses these planes as planar elements, as we can observe:

Definition 9. Let *S* be a free–form solid whose surface is represented by a set of TBP's, let $b(\tau)$ be a patch of this set, and let t_0 be its base triangle. A *free–form cell* of *S* is defined as the intersection of the half–spaces determined by $b(\tau)$, its three neighbouring TBP's (t_1, t_2 and t_3 being their base triangles), and the three planes associated $\pi(t_0, t_1)$, $\pi(t_0, t_2)$ and $\pi(t_0, t_3)$. ♠

Figure 6.b shows an example of ffc. The associated planes are drawn using four edges polygons; the base triangle belongs to the initial triangulation of a solid. Figure 6.a does not show the neighbouring TBP's in order to give a better view of the picture.

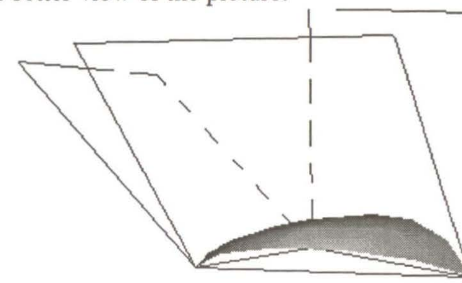

Figure 6.a. Partial view of a ffc

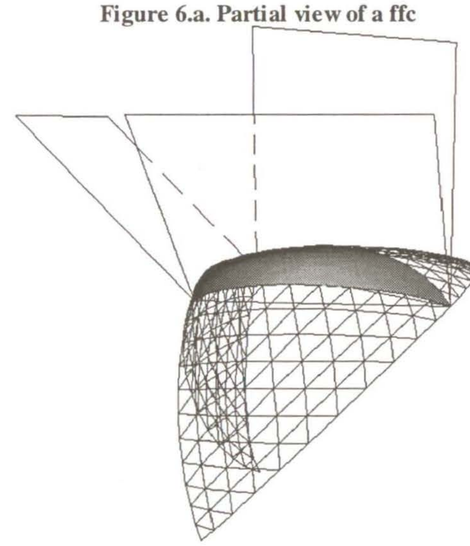

Figure 6.b. Free-form cell

We did not choose the planes that contain the curved edges of the TBP's as the associated planes because of the following: when we described the TBP's, we said that the edges of every patch are Bézier curves defined by the control points of the edges of the control point net of the patch. As we can observe in figure 7, it is possible to find TBP's whose control points from one or more edges of the control point net do not lie in the same plane; therefore, the Bézier curves that these points define may not lie in a plane, and consequently, the choice of the planes that contain the the curved edges of the TBP's is not appropriate, because these planes do not exist always. This is the reason to use the planes explained in definition 8. In figure 7, the plane that contains three control points of an edge is drawn with a black line, and the remaining point lies outside that plane, showing that the corresponding curved edge does not lie in a plane.

If the TBP's passes through the base triangles, then we consider each connected component as a different ffc, as shown in figure 8.

To calculate the sign associated to a ffc, we make use of the plane that contains the base triangle (suppose the vertices of the triangle given in counterclockwise). If the ffc is in the possitive half–space defined by that plane, its sign is +1, and –1 otherwise (0 if the ffc and the base triangle coincide). See figure 9.

Figure 7. Top vision of a TBP

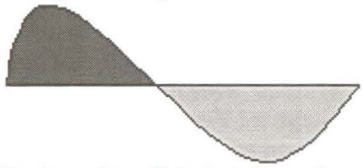

Figure 8. 2D view of two ffc's that share the same patch

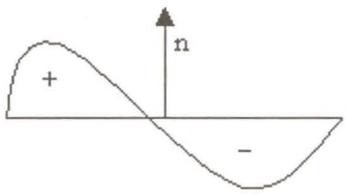

Figure 9. Sign of two ffc's

The ffc defined this way verifies the three conditions mentioned before:

1. As we can see, the ffc's are closed and connected sets of points in $\mathbb{R}^3$.
2. The second condition does not suppose a problem, because we calculate the sign for a point using the plane that contains the base triangle instead of the implicit equation of the patch. Moreover, the algorithm used to construct the patches achieves that there are no self-intersections in the parametric domain [0.0, 1.0] of the patch.
3. The third condition is verified by this kind of ffc too, as it can be seen in the figures. The vertices holding this condition are the three vertices of the base triangle of each ffc.

4. FREE-FORM SOLID REPRESENTATION

As it can be infered from definition 4, several ESC's may correspond to the same solid. It is necessary to select one of them in order to represent the set of equivalent ESC's for a given free-form solid. For this reason, the *normal* ESC's are defined this way:

Definition 10. Let δ be an ESC. We will say that δ is *normal* if and only if it verifies:

$$f_\delta(Q)=1, \quad \forall Q\in FF_\delta \qquad ♠$$

To construct the normal ESC that uniquely determines and represents a given 3-dimensional free-form solid, we need to obtain the extended cells (simplices and ffc's) that compose it, and the associated coefficient for each one. This process starts from an initial triangulation of the solid, and continues as follows:

- Each triangle t_i from the initial triangulation determines an unique *original simplex* (tetrahedron), formed by its three vertexes and the origin **O**.
- Each triangle t_i from the initial triangulation determines a unique TBP, and a ffc related to it and its neighbourhood, as described in definition 9.

Figure 10 shows an example of the extended cells obtained for a triangle from a solid triangulation. The triangle and the original tetrahedron are drawn in red. The ffc is drawn like the one in figure 6b.

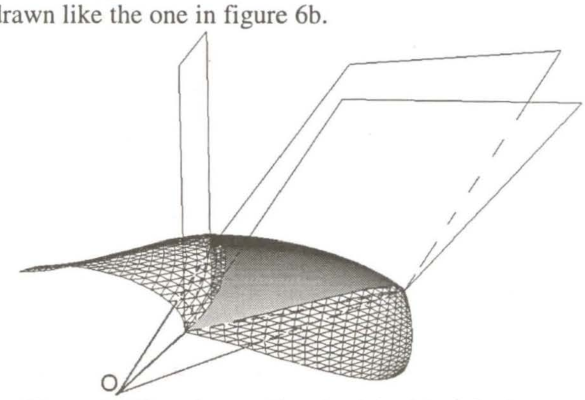

Figure 10. Free-form cell and original tetrahedron

The associated coefficient α_i for each ffc is obtained as follows:

- For each original simplex, the coefficient is the sign of its signed volume. See [Feito98].
- For each ffc, its associated coefficient is its sign, as described above.

The fourth vertex to construct the tetrahedra may be any point in $\mathbb{R}^3$. The reason for the choice of the coordinate origin is the efficiency in the coefficient calculation. The election of any other point in $\mathbb{R}^3$ does not affect the fact that the chain obtained is a normal chain. A possible subject to study is the influence of the fourth vertex election in the complexity of the ESC obtained, and an algorithm to elect the best origin for each chain.

Once we have described the method to obtain the normal ESC for a given free-form solid, it is necessary to establish the equivalence between the chain and the solid. Thus, the following theorem is enunciated:

Theorem 1. Let FF be a free-form solid with an initial triangulation formed by n triangles $t_1, t_2, ..., t_n$ (triangle vertices ordered counterclockwise), and $b_i(\tau)$ be the TBP constructed for t_i. Let S_i be the original simplex (tetrahedron) determined by the origin **O** and the triangle t_i, whose associated coefficient is s_i=*sign of* S_i, and let ffc_i be the ffc defined by the patch $b_i(\tau)$, the triangle t_i, its neighbouring elements and the associated planes, whose associated coefficient is a_i=*sign of* ffc_i. Then:

$$FF=FF_\delta \quad where \quad \delta=\sum_{i=1}^{n}\left(s_i\cdot S_i+a_i\cdot ffc_i\right) \quad \textbf{(1.}$$

Proof. We need to prove that any point **Q** from FF belongs to FF_δ, and vice versa. As the solids are closed and bounded, we only need to prove the theorem for the interior points of them. To do this, let us consider the half line from the origin **O** that passes through **Q** and goes outside the solid. Applying Jordan's theorem, the half line will have an odd number of intersections with the surface of the solid **starting from Q** if the point is in the solid, and an even number of intersections (including 0) otherwise. First, let us prove that $FF \subseteq FF_\delta$. Suppose that **Q** is in the solid.

If there were no ffc's, then the half line would have p intersections with triangles that define original simplices with negative coefficient (transitions from outside to inside the solid), and $p+1$ intersections with triangles that define original simplices with possitive coefficient (transitions from inside to outside the solid). Therefore, **Q** belongs to p original simplices S_i with coefficients $s_i=-1$, and to $p+1$ original simplices S_i with coefficients $s_i=+1$. Applying the associated function for the solid FF_δ, results: $f_\delta(Q)=(p+1)-p=1$; so **Q** belongs to the normal simplicial chain (not extended) δ. As we use regularized operations, the points that are shared by two simplices are not a problem, because in every neighbourhood of them there will exist an accumulation point of FF_δ.

If we consider the ffc's, there are two possible situations: **Q** may belong to one ffc or may not:

- If the point is not in any ffc, the half line from **Q** will intersect zero or an even number of times the border elements of the ffc, and there will be no contribution from ffc's to the sum in (1). As the result of this sum was already 1, **Q** belongs to the solid.
- If the point is in one ffc, the half line from **Q** will intersect an odd number of times the border elements of the ffc, and there will be a term from the ffc in the sum. If the point is outside the solid defined by the simplices, and the coefficient of the ffc's is equal to +1, then the point will be in the solid.

Again, the inclusion of points shared by several extended cells is solved by the use of regularized operations, as we have seen above.

Once we have proved that $FF \subseteq FF_\delta$, let us prove the opposite case; i.e. $FF_\delta \subseteq FF$. To do this, we are going to demonstrate that every point **Q** from outside FF is not in FF_δ. Let us consider the half line from the origin that passes through **Q**. This half line will intersect the surface of the solid an even number of times. As we can consider the curved zones of the solid as planar faces (as small as we want), then **Q** will belong to zero or an even number of original simplices, half of them with associated coefficient equal to +1 (one for each transition from the exterior to the interior of the solid) and the other half with coefficient –1 (one for each transition from the interior to the exterior of the solid). Therefore, the value of the associated function will be $f_\delta(Q)=0$, and **Q** will not belong to the solid FF_δ. ♣

5. BOOLEAN OPERATIONS

The use of ESC's allows us to simplify the Boolean operations with solids, because we can apply the *divide and conquer* approach by decomposing the operation with whole solids in simpler operations with the extended cells that form them, and combining the results in a sum. This makes the operations be done homogeneously, avoiding particular cases. In contrast to other models, it is not necessary to change the representation of the solids to operate with them. In [Feito98], this is applied with simplicial chains, and in [Ruiz01a] the operations with ESC's are explained. Here we review them, adapting the theorems when necessary.

In order to represent Boolean operations, we make use of the intersection of extended cells. The result of the intersection of two extended cells E and E' may be other extended cell, but usually it is a set of points that can be not connected. However, this set can be decomposed (more or less easily) in extended cells, and therefore we can associate a normal ESC to the intersection. Let us call this chain $ExCell\ (E \cap E')$. So, the representation of Boolean operations between free–form solids expressed as ESC's is given by the following theorems:

5.1 Intersection

First of all, let us study the intersection of solids:

Theorem 2. Let FF_1 and FF_2 be two free–form solids, and let δ_1 and δ_2 be their associated normal ESC's, expressed as:

$$\delta_1=\sum_{i=1}^{n} a_i \cdot E_i; \qquad \delta_2=\sum_{j=1}^{m} b_j \cdot F_j;$$

then, the associated normal ESC for the solid obtained as the intersection $FF_\delta = FF_1 \cap FF_2$ is:

$$\delta=\sum_{i=1}^{n} \sum_{j=1}^{m} (a_i \cdot b_j) \cdot ExCell\left(E_i \cap F_j\right); \qquad (2.$$

Proof. We have to prove that any point **Q** from FF_δ belongs to the intersection of FF_1 and FF_2 and vice versa. As we have mentioned before, it is only necessary to demonstrate it for the interior points of the solids, because the solids are closed and bounded, and we are working with regularized operations.

Let **Q** be an interior point of $FF_1 \cap FF_2$. Therefore, it belongs to both FF_1 and FF_2. If **Q** belongs to FF_1, there will be q negative cells from FF_1 that will contain the point, and $q+1$ possitive cells from FF_1 to which the point will belong. The same will occur with FF_2: m negative cells will contain the point, and so will $m+1$ possitive cells.

When computing the associated value for the chain in (2), the possible combinations are:

- E_i and F_j are possitive. This will happen $(q+1)\cdot(m+1)$ times, and the resulting coefficient will be possitive.
- E_i possitive and F_j negative. This will happen $(q+1)\cdot m$ times, and the resulting coefficient will be negative.
- E_i negative and F_j possitive. This will happen $q\cdot(m+1)$ times, and the resulting coefficient will be negative.
- E_i and F_j negative. This will happen $q\cdot m$ times, and the resulting coefficient will be possitive.

Therefore, the result of the sum will be:

$$(q+1)\cdot(m+1)-(q+1)\cdot m-q\cdot(m+1)+q\cdot m=1;$$

and as a consequence, **Q** belongs to FF_δ.

Let us consider now the situation of a point **Q** that is in FF_1, but it does not belong to FF_2. Similarly to the previous explanation, there will be q negative cells and $q+1$ possitive cells from FF_1 that will contain the point, and so will m negative cells and m possitive cells from FF_2. When computing the associated value for the chain in (2), the possible combinations are:

- E_i and F_j are possitive. This will happen $(q+1)\cdot m$ times, and the resulting coefficient will be possitive.
- E_i possitive and F_j negative. This will happen $(q+1)\cdot m$ times, and the resulting coefficient will be negative.
- E_i negative and F_j possitive. This will happen $q\cdot m$ times, and the resulting coefficient will be negative.
- E_i and F_j negative. This will happen $q\cdot m$ times, and the resulting coefficient will be possitive.

Therefore, the result of the sum will be:

$$(q+1)\cdot m-(q+1)\cdot m-q\cdot m+q\cdot m=0;$$

and as a consequence, **Q** does not belong to FF_δ.

The same reasoning can be applied to the points that belong to FF2 and are not in FF1, and to points that do not belong to any of the solids. In both situations, the result of the sum will be 0. ♣

The application of this theorem allows us to solve the intersection of solids starting from the intersection of extended cells.

5.2 Union

Next, let us focus our attention on the union of solids.

Theorem 3. Let FF_1 and FF_2 be two free–form solids, let δ_1 and δ_2 be their associated normal ESC's, expressed as:

$$\delta_1=\sum_{i=1}^{n} a_i\cdot E_i;\qquad \delta_2=\sum_{j=1}^{m} b_j\cdot F_j;$$

and let $\delta_{FF1\cap FF2}$ be the associated normal ESC for the intersection of FF_1 and FF_2. Then, the associated normal ESC for the solid obtained as the union $FF_\delta = FF_1 \cup FF_2$ is:

$$\delta=\delta_1+\delta_2-\delta_{FF_1\cap FF_2}; \qquad (3.$$

Proof. As we have explained before, we are going to prove the theorem only for interior points. For every point **Q** there will be four possibilities:

- **Q** does not belong to either FF_1 or FF_2. Therefore, it will not belong to the intersection, and the associated value for the chain in (3) will be 0.
- **Q** belongs only to FF_1. Therefore, the associated value for the chain in (3) will be:

$$f_\delta(Q)=1+0-0=1;$$

  and as a consequence, **Q** belongs to FF_δ.
- **Q** belongs only to FF_2. Therefore, the associated value for the chain in (3) will be:

$$f_\delta(Q)=0+1-0=1;$$

  and as a consequence, **Q** belongs to FF_δ.
- **Q** belongs to the intersection FF_1 O FF_2. Therefore, the associated value for the chain in (3) will be:

$$f_\delta(Q)=1+1-1=1;$$

  and as a consequence, **Q** belongs to FF_δ. ♣

5.3 Complementary

Now let us enunciate the theorem applied to the complementation of solids.

Theorem 4. Let FF be a free–form solid, and let δ be its associated normal ESC's, expressed as:

$$\delta=\sum_{i=1}^{n} a_i\cdot E_i;$$

then, the associated normal ESC for the solid obtained as the complementary FF^c is:

$$\delta^c=\delta_R-\delta; \qquad (4.$$

where δ_R stands for the chain whose associated function equals 1 for every point in $\mathbb{R}^d$, d being the dimension of the solid.

Proof. Because of the same reason explained before, we are going to prove the theorem for interior points.

Let **Q** be a point of FF^c. **Q** does not belong to FF, but to $FF_{\delta R}$. Therefore, it belongs to $FF_{\delta c}$, because the associated value for the chain in (4) is:

$$f_\delta(Q)=1-0=1;$$

If **Q** does not belong to FF^c, then it is in FF and in $FF_{\delta R}$. Therefore, it does not belong to $FF_{\delta c}$, because the associated value for the chain in (4) is:

$$f_\delta(Q)=1-1=0;$$ ♣

In practice, δ_R can be implemented as a d–dimensional cube that contains the whole scene.

5.4 Difference

Finally, we will explain the theorem for the difference of solids.

Theorem 5. Let FF_1 and FF_2 two free–form solids, and let δ_1 and δ_2 be their associated normal ESC's, expressed as:

$$\delta_1=\sum_{i=1}^{n} a_i\cdot E_i;\qquad \delta_2=\sum_{j=1}^{m} b_j\cdot F_j;$$

then, the associated normal ESC for the solid obtained as the difference $FF_\delta = FF_1 - FF_2$ is:

$$\delta=\delta_1-\delta_2; \qquad (5.$$

Proof. Considering only interior points, let **Q** be a point that belongs to the difference solid. So, it is in FF_1, but not in FF_2. Therefore, the associated value for the chain in (5) will be:

$$f_\delta(Q)=1-0=1;$$

and as a consequence, **Q** belongs to FF_δ.

Similarly, if **Q** does not belong to the difference solid, there are three possibilities:

- **Q** does not belong to both FF_1 and FF_2. The associated value for the chain in (5) is:

  $$f_\delta(Q)=0-0=0;$$

  therefore, **Q** does not belong to FF_δ.
- **Q** belongs to both FF_1 and FF_2. The associated value for the chain in (5) is:

  $$f_\delta(Q)=1-1=0;$$

  therefore, **Q** does not belong to FF_δ.
- **Q** belongs only to FF_2. The associated value for the chain in (5) is:

  $$f_\delta(Q)=0-1=-1;$$

  therefore, **Q** does not belong to FF_δ. ♣

To make the associated value for the chain be 1 for the points in the difference solid and 0 in the rest, we can redefine the ESC associated to the difference as:

$$\delta=\delta_1-\delta_{FF_1\cap FF_2};$$

The value for the chain in the last case of the previous demonstration will be now:

$$f_\delta(Q)=0-0=0;$$

and **Q** does not belong to the difference again.

We have used algebraic operations to define the boolean ones. As these operations are accumulative, it is not necessary to use only disjoint cells, but also we can handl. non–disjoint cells. Unlike CSG systems, the calculations are made on very simple solids (the cells), so the special cases are minimum and well–known.

As a sample, we present figures 11 to 17 in which we show two free–form solids (a stomach and an abstract solid), and the result of the boolean operations between them, using a regular point mesh to visualize the resulting solid in each situation. For each operation, we show a visualization of the points included int the complete ESC, and an intermediate image, showing the points in a subset of the resulting chain; in these intermediate images, we use red color to mark points that are in the subset, but do not belong to the final resulting chain.

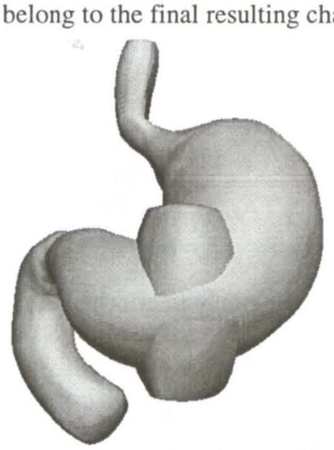

Figure 11. Two free-form solids

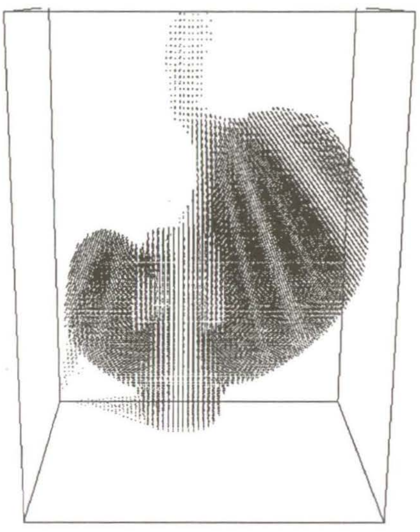

Figure 12. Intermediate result of the union of solids from figure 11

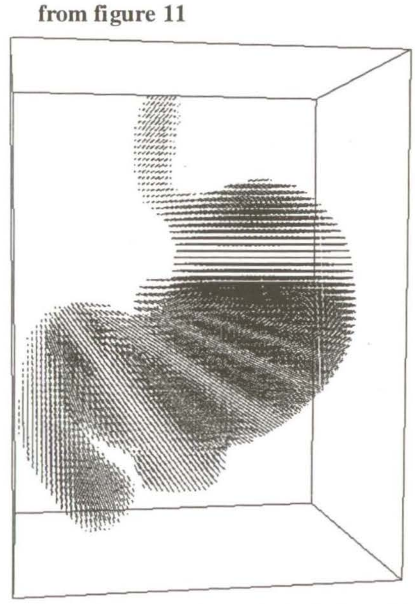

Figure 13. Result of the union of solids from figure 11

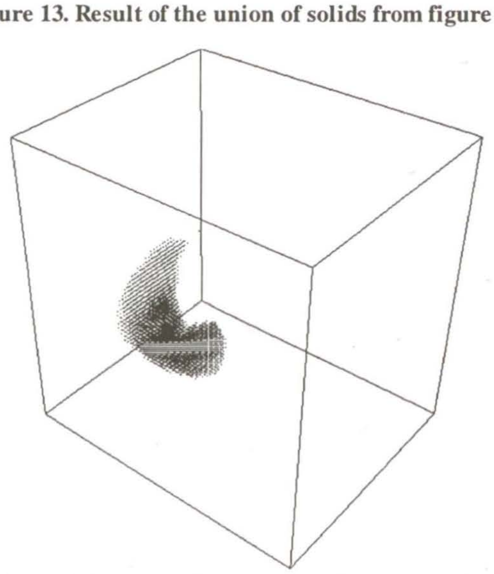

Figure 14. Intermediate result of the intersection of solids from figure 11

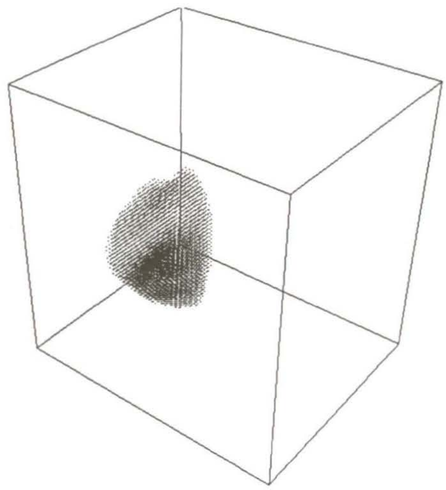

Figure 15. Result of the intersection of solids from figure 11

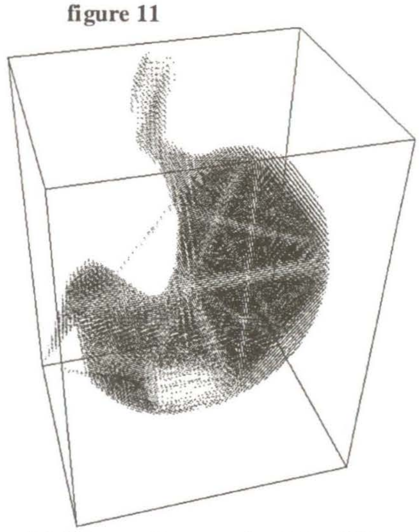

Figure 16. Intermediate result of the difference of solids from figure 11

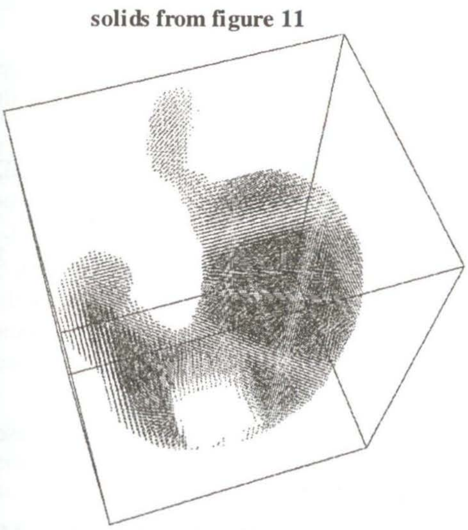

Figure 17. Result of the difference of solids from figure 11

6. CONCLUSIONS

In this paper we have presented a mathematical method to represent and manage free–form solids based on the concepts of free–form cell (ffc) and extended simplicial chain (ESC) previously presented in [Ruiz99, Ruiz01a]. The model introduces a new kind of ffc based on triangular Bézier patches that verifies the properties for general ffc's, and adapts the definitions and theorems to the specific case of solids whose surface is composed by this kind of patches.

The model allows us to represent in an uniform way both free–form solids and Boolean operations with them, without particular cases nor foreign elements to the model because it is based on algebraic operations that are accumulative, and allows us to handle both disjoint and non–disjoint cells.

As mentioned before, the use of ESC's is independent of both the dimensions and type of free–form elements used to define the ffc's, and although the solid decomposition may involve a high number of operations with the cells, these operations are simpler, and in many cases the implementation can be optimized, fastening the management of the solids.

The data structures necessary to implement this model are not much different from the characteristic ones used for B–rep modelling, so this model present an easy and robust method to work with free–form solids in 3D. In fact, a point–in–solid test based on this model [García01] has been already successfully implemented (see figures 12–19 for examples of the results obtained). As we can see in figure 20, most of the times the test does not need to calculate as many intersections ray–boundary of the solid as the classic algorithms based on Jordan's theorem (in figure 20 we can see the results for three different solids; the graph shows the evolution of the number of intersections to calculate vs. the number of points used in the test). So, the model has proved to be useful in at least one of the classical problems in geometric modelling.

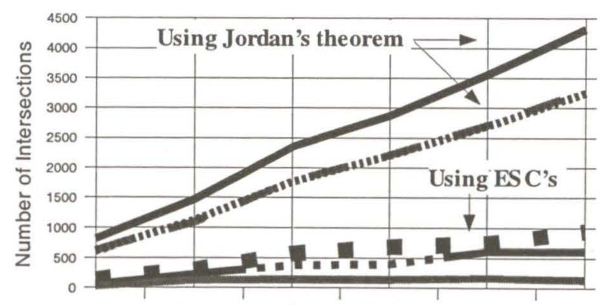

Figure 20. Graphic comparison between the number of intersections ray-surface calculated with Jordan's theorem and ESC's. See [García01] for more details

The *f(Q)* function of an ESC (from a solid or a boolean combination of several solids) can be easily implemented because it is based in the inclusion test for each extended cell solved by methods like the ones in [García01]. Therefore, it is possible to develop an algorithm for the direct visualization of the solid associated with an ESC using ray–casting methods on the points *Q* such that *f(Q) = 1*. In fact, this is the subject of our current work.

7. ACKNOWLEDGEMENTS

This work has been partially granted by the Ministry of Science and Technology of Spain and the European Union by means of the ERDF funds, under the research project TIC2001–2099–C03–03

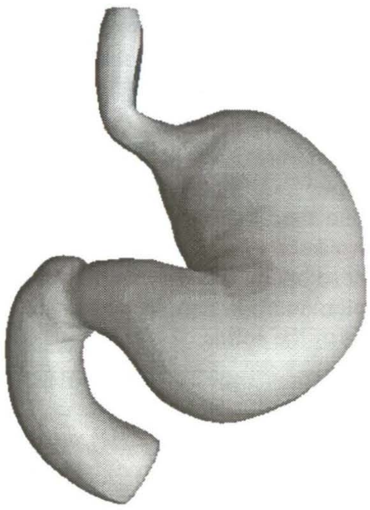

Figure 18. One of the solids used in the tests for our point inclusion algorithm

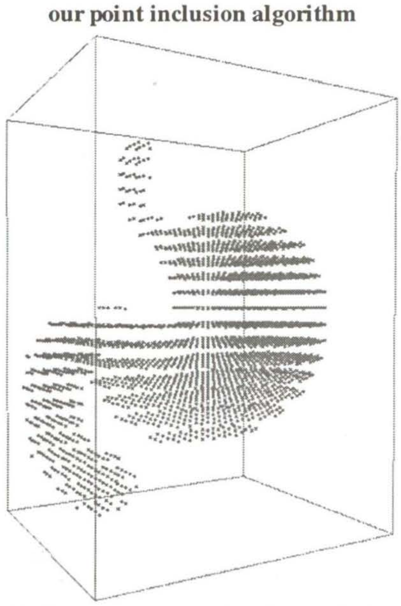

Figure 19. Points in the solid of figure 18 from a regular set of 25x25x25 points.

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