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Poster

Functional Maps on Product Manifolds

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Abstract

We consider the tasks of representing, analyzing and manipulating maps between shapes. We model maps as densities over the product manifold of the input shapes; these densities can be treated as scalar functions and therefore are manipulable using the language of signal processing on manifolds. Being a manifold itself, the product space endows the set of maps with a geometry of its own, which we exploit to define map operations in the spectral domain. To apply these ideas in practice, we introduce localized spectral analysis of the product manifold as a novel tool for map processing.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Shape Analysis, 3D Shape Matching, Geometric Modeling

1. Introduction

A modern theme in shape correspondence involves the *representation* of a map from one shape to another. While an obvious representation maintains source and target points, this is not the only option. Our paper is related to two frameworks developed for establishing correspondence between shapes: optimization on *product manifolds* and *functional maps*.

Motivation and contribution. We advocate posing correspondence in terms of functions on the product manifold of the source and target. A motivating observation is that functional maps approximate a distribution representing the correspondence in the product space as a linear combination of *separable* basis functions. This distribution is supported on a manifold with a dimension *lower* than the product space. Consequently, most of the support of the basis functions is wasted on "empty" regions of the product space. This viewpoint suggests new techniques to represent and approximate mappings directly on the product. Reasoning about the product manifold leads to compact, understandable bases for map design that focus resolution in the part of the product most relevant to a correspondence task. One of such means is the construction of *inseparable* bases as localized harmonics on the product manifold.

2. Map representation on the product manifold

Given two manifolds $(\mathcal{M}, g_{\mathcal{M}}), (\mathcal{N}, g_{\mathcal{N}})$ of dimension $d_{\mathcal{M}}$ and $d_{\mathcal{N}}$, respectively, their product $(\mathcal{M}\times\mathcal{N}, g_{\mathcal{M}} \oplus g_{\mathcal{N}})$ is a manifold of dimension $d_{\mathcal{M}} + d_{\mathcal{N}}$. All properties, like the metric tensor or the spectral decomposition, can be derived from the original manifolds properties by simple algebraic operations [Cha06].

Functional Maps are linear operators $T_F : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{N})$ between functional spaces on $\mathcal{M}, \mathcal{N}$, where $T_F(f) = f \circ T^{-1}$ is based upon a bijection $T : \mathcal{M} \to \mathcal{N}$.

Soft functional maps. We introduce a "soft" generalization of functional maps. For soft maps $\tilde{\mu} : \mathcal{M} \to \text{Prob}(\mathcal{N})$ [SNB*12] with associated density $\mu \in L^1(\mathcal{M}\times\mathcal{N})$, we define a *soft functional map* T_μ as the expectation

$$T_\mu(g)(x) = \int_{\mathcal{N}} g(y)\mu(x,y)\,\mathrm{d}y\,. \tag{1}$$

A connection can be derived between functional maps and expanding soft map measures in the Laplace–Beltrami basis:

Theorem 1 (Equivalence) Let $T_\mu : \mathcal{F}(\mathcal{N}) \to \mathcal{F}(\mathcal{M})$ be a soft functional map (1) with underlying density $\mu \in L^1(\mathcal{M}\times\mathcal{N})$. Further, let $c_{ij} = \langle \phi_i, T_\mu(\psi_j)\rangle_{\mathcal{M}}$ be the matrix coefficients of T_μ in the orthogonal bases $\{\phi_i\}_{i\geq 1} \subseteq \mathcal{F}(\mathcal{M}), \{\psi_j\}_{j\geq 1} \subseteq \mathcal{F}(\mathcal{N})$, and let $p_{ij} = \langle \phi_i \wedge \psi_j, \mu\rangle_{\mathcal{M}\times\mathcal{N}}$ be the expansion coefficients of μ in the product basis $\{\phi_i \wedge \psi_j\}_{i,j}$, such that $\mu = \sum_{ij}(\phi_i \wedge \psi_j)p_{ij}$. Then, $c_{ij} = p_{ij}$ for all i, j.

Spectral representation. Consider the order-k, band-limited ap-

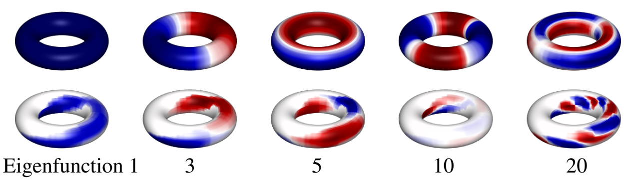

Figure 1: *Examples of basis functions on the product manifold (here visualized as a torus embedded in $\mathbb{R}^3$) of two 1D shapes. We plot a few standard LB eigenfunctions (top row) and localized manifold harmonics (bottom row). The first basis function in the bottom row also indicates the used region. In the color scheme blue denotes small values, red large values and white is zero.*

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proximation of μ:
$$\mu \approx \sum_{\ell=1}^{k} \xi_\ell p_\ell \,, \quad (2)$$

where each ξ_ℓ is an eigenfunction of $\Delta_{\mathcal{M}\times\mathcal{N}}$ which uniquely identifies a pair of eigenfunctions ϕ_i, ψ_j on $\mathcal{M}$ and $\mathcal{N}$. According to Theorem 1, the expansion coefficients p_ℓ are exactly those appearing in the functional map $\mathbf{C}$, when it is expressed in the Laplacian eigenbases of $\mathcal{M}$ and $\mathcal{N}$ as originally proposed in [OBCS*12].

Truncation. The product eigenfunctions ξ_ℓ appearing in the summation (2) are associated to the product eigenvalues $\alpha_i + \beta_j$, which are ordered non-decreasingly. In contrast, in [OBCS*12] it was proposed to truncate the eigenvalues to $i = 1, \ldots, k_\mathcal{M}$ and $j = 1, \ldots, k_\mathcal{N}$, where indices i and j follow the non-decreasing order of the eigenvalue sequences α_i and β_j *separately*. Due to the different ordering the eigenfunctions ϕ_i, ψ_j involved in the approximation (2) of μ are not necessarily those involved in the construction of a standard Functional Map $\mathbf{C}$. In the former case we operate with a reduced basis directly on $\mathcal{M}\times\mathcal{N}$, while in the latter case we consider two reduced bases on $\mathcal{M}$ and $\mathcal{N}$ *independently*. See Figure 2.

Localized spectral encoding. Theorem 1 establishes the equivalence between the soft functional map T_μ representation coefficients c_{ij} in the bases $\{\phi_i\}_{i\geq 1}$ and $\{\psi_j\}_{j\geq 1}$ and the coefficients p_ℓ of the underlying density μ Fourier series (2) in the eigenbasis $\{\xi_\ell\}_{\ell\geq 1} \subseteq \mathcal{F}(\mathcal{M}\times\mathcal{N})$ of the product manifold Laplacian $\Delta_{\mathcal{M}\times\mathcal{N}}$. This equivalence directly stems from ξ_ℓ's having the separable form $\phi_i \wedge \psi_j$. It may be advantageous to consider different orthonormal bases on $\mathcal{M}\times\mathcal{N}$ that are not necessarily separable. In particular, we observe that μ tends to be localized on the product manifold $\mathcal{M}\times\mathcal{N}$, and thus the standard outer product basis is extremely wasteful as it is supported on the entire $\mathcal{M}\times\mathcal{N}$.

A better alternative is the use of *localized manifold harmonics* [MRCB17] on the support of μ which is *no more separable*, i.e., the functions are not in general expressible as outer products of functions defined on the originating domains but the size of such problems is huge, and despite their extreme sparsity, computationally expensive. As an alternative, we consider a *patch* $\mathcal{P}\subset\mathcal{M}\times\mathcal{N}$ of the product manifold, and define the eigenproblem

$$\begin{array}{ll} \Delta_\mathcal{P}\bar{\xi}_\ell(x,y) = \gamma_\ell \bar{\xi}_\ell(x,y) & (x,y)\in \mathrm{int}(\mathcal{P}) \\ \bar{\xi}_\ell(x,y) = 0 & (x,y)\in \partial\mathcal{P} \end{array} \quad (3)$$

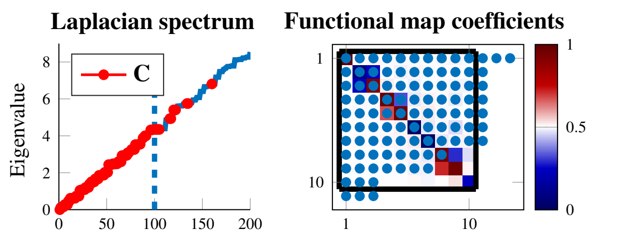

Figure 2: Left*: The* $k = 100$ *frequencies involved in the* 10×10 *functional map* $\mathbf{C}$ *correspond to an irregular sampling of the spectrum of the product manifold.* Right*: Only some of the* c_{ij} *appear among the first k coefficients in the product eigenbasis. Here* $\mathbf{C}$ *is framed in black, while the blue dots identify the coefficients* p_{ij}.

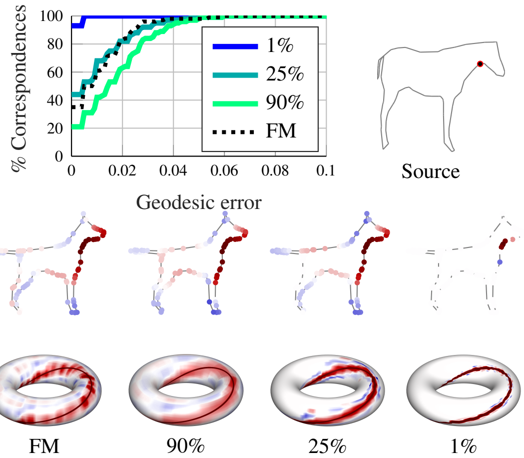

Figure 3: *Product space approximation of the correspondence between one-dimensional shapes with* $k = 100$ *basis functions. Correspondences can be iteratively refined by reducing the patch area* $\mathcal{P}$ *(1%, 25% and 90% of the total product manifold area shown here). Top left: accuracy of the correspondence increases as the product space basis becomes more localized. Middle row: image of a delta function by the functional maps. Bottom: Ground-truth (curve) and its approximation in the product bases with a varying degree of localization. Separable basis (FM) is shown as a reference.*

of the *product patch Laplacian* $\Delta_\mathcal{P}$ with Dirichlet boundary conditions. This choice reduces the complexity to the size of the patch instead of the entire product manifold.

3. Discussion

We proposed the adoption of (inseparable) localized harmonics for compactly encoding correspondences while ensuring minimal energy dispersion. Our theoretical work suggests a new perspective on properties of the correspondence manifold as well as new ways to pose algorithmic design for map inference and processing. We hope that this work promotes further research on algorithmic design for map inference and processing, while casting new light on useful properties of the correspondence manifold.

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