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Poster

Solving variational problems using nonlinear rotation-invariant coordinates

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Abstract

We consider Nonlinear Rotation-Invariant Coordinates (NRIC) representing triangle meshes with fixed combinatorics as a vector stacking all edge lengths and dihedral angles. Previously, conditions for the existence of vertex positions matching given NRIC have been established. We develop the machinery needed to use NRIC for solving geometric optimization problems. Moreover, we introduce a fast and robust algorithm that reconstructs vertex positions from close-to integrable NRIC. Our experiments underline that NRIC-based optimization is especially effective for near-isometric problems.

1. Introduction

Variational problems are at the core of many applications in geometry processing. Here we consider Nonlinear Rotation-Invariant Coordinates (NRIC) for solving them on triangle meshes. These coordinates offer benefits such as their inherent invariance to rigid transformations and their natural occurrence in discrete deformation energies. Prior work on shape interpolation by Winkler *et al.* [WDAH10] and Fröhlich and Botsch [FB11] showed that linear blending of the NRIC of a set of shapes already yields alluring nonlinear deformations. However, since in general vertex positions that realize given edge lengths and dihedral angles may not exist, these methods rely on optimization in the space of vertex positions. In contrast, we use constrained optimization in NRIC to avoid this.

2. The NRIC manifold of edge lengths and dihedral angles

We consider the *fixed* connectivity graph $\mathcal{G} = (\mathcal{V}, \mathcal{F}, \mathcal{E})$ of a triangular surface, where $\mathcal{V}$ is an index set of vertices, $\mathcal{F}$ the set of faces, and $\mathcal{E}$ the set of edges. Then an admissible embedded mesh is given by the image of an immersion and we define $\mathcal{N} := \{\mathbf{I}(\mathcal{V}) \mid \mathbf{I}\colon \mathcal{V} \to \mathbb{R}^3 \text{ immersion}\} \subset \mathbb{R}^{3|\mathcal{V}|}$ which we denote the space of *discrete surfaces*. We aim at describing $\mathcal{N}$ by the *Nonlinear Rotation-Invariant Coordinates (NRIC)*, *i.e.* the vectors of edge lengths and dihedral angles. To this end, we first define $Z\colon \mathcal{N} \to \mathbb{R}^{2|\mathcal{E}|}$, $X \mapsto (l(X), \theta(X))$ mapping a discrete surface to its edge lengths and dihedral angles. Furthermore, we will consider $z = (l, \theta) \in \mathbb{R}^{2|\mathcal{E}|}$ a tuple of lengths and angles, regardless of the fact whether it belongs to a discrete surface or not. Then the image of Z defines the NRIC manifold $\mathcal{M} := Z(\mathcal{N})$ of tuples that actually do. Next, we will explain how $\mathcal{M}$ can be defined as an implicit submanifold via a set of integrability conditions.

Discrete integrability conditions. In this section, we briefly review the integrability conditions introduced by Wang *et al.* [WLT12] for simply connected surfaces. The first straight forward necessary condition is fulfilling the *triangle inequalities* for each face $f \in \mathcal{F}$. Furthermore, we define for adjacent faces f_i and f_j the transition rotation $R_{ij} := R_0(\theta_i) R_2(\gamma_j)$, where γ_j is the interior angle at v in f_j and θ_i is the dihedral angle at the common edge. Then one can phrase the integrability condition as

$$\mathcal{I}_v(l, \theta) := \prod_{i=0}^{n_v - 1} R_{i,(i+1) \bmod n_v} \stackrel{!}{=} \mathbb{1} \tag{I}$$

for the n_v-loop of faces around all *interior* vertices $v \in \mathcal{V}_0$. Wang *et al*. [WLT12] proved for simply connected surfaces the sufficiency of the two conditions for the existence of a matching discrete surface.

Embedding as an implicit submanifold. To reduce the number of constraints, we choose Euler angles $\Lambda(Q) \in \mathbb{R}^3$ in $x-y-z$ orientation for every rotation matrix $Q \in SO(3)$ and reformulate (I) as $\Lambda(\mathcal{I}_v(z)) = 0$ for all interior vertices $v \in \mathcal{V}_0$. This yields a vector-valued constraint functional $G\colon \mathbb{R}^{2|\mathcal{E}|} \to \mathbb{R}^{3|\mathcal{V}_0|}$, where $G(z) = (G_v(z))_{v \in \mathcal{V}_0} = (\Lambda(\mathcal{I}_v(z)))_{v \in \mathcal{V}_0}$. To condense the triangle inequality and the integrability (I) in a single condition, we set $G_v(z) = \infty$ whenever the triangle inequality is violated for a face f with $v \in f$. This allows one to rewrite the NRIC manifold as

$$\mathcal{M} = \left\{ z \in \mathbb{R}^{2|\mathcal{E}|} \,\middle|\, G(z) = 0 \right\}. \tag{1}$$

This implicit formulation allows to compute the tangent space of $\mathcal{M}$ as $T_z\mathcal{M} = \ker DG(z)$. The gradient of G_v can be computed in $O(n_v)$ cost exploiting its structure via the chain and product rule for matrix-valued maps. Furthermore, it is sparse and has only $O(n_v)$ non-vanishing entries. With the tangent space at hand, one can for

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instance easily verify the *infinitesimal* rigidity of $z \in \mathcal{M}$ by checking for tangent vectors with zero length component, cf. Fig. 1.

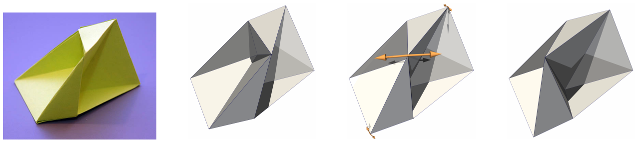

Figure 1: *The tangent space reveals an infinitesimal isometric variation at the classical Steffen's polyhedron (middle). Indeed, extrapolating in this positive (left) resp. negative (right) direction (solely in the θ component) allows for isometric deformations. Photograph courtesy of Laszlo Bardos from cutoutfoldup.com.*

3. Variational problems on the manifold

The quest for geometrically optimal, discrete surfaces often leads to variational problems. However, in many applications, the corresponding objective functional can naturally be formulated in our coordinates, thus on the NRIC manifold (1), and its first and second variation can easily be computed. To this end, one aims at solving a constrained optimization problem, *i.e.* minimize an objective functional $E\colon \mathbb{R}^{2|\mathcal{E}|} \to \mathbb{R}$ subject to the nonlinear constraint $G(z) = 0$.

A simple example of an objective functional is given by a weighted (squared) distance measure $\mathcal{W}$ to some given z^* on the linear space $\mathbb{R}^{2|\mathcal{E}|}$, *i.e.* $E(z) = \mathcal{W}[z, z^*]$, where

$$\mathcal{W}[z,z^*] = \sum_{e\in\mathcal{E}} \alpha_e \|l_e - l_e^*\|^2 + \delta^2 \sum_{e\in\mathcal{E}} \beta_e \|\theta_e - \theta_e^*\|^2 . \qquad (2)$$

This can be seen as an elastic *deformation energy*. In fact, almost the same model has been used in [GHDS03] to define the *Discrete Shells* energy for physical simulations based on nodal positions.

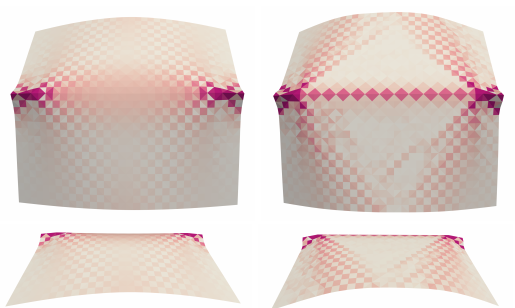

Figure 2: *Paper folding with local constraints for dihedral angle: simulation in vertex space (left) leads to small isometry violations whereas the result in NRIC is fully isometric (right). Triangle-averaged mean curvature is shown as color map* 0 ≥ 0.015.

In our experiments, this setup was particularly effective for problems involving near-isometric deformations. For instance, we simulated folding a flat sheet of paper by imposing isometry constraints $l = l^*$ and angles constraints $\theta_i \neq 0$ for certain $i \in I \subset \mathcal{E}$, cf. Fig. 2. Furthermore, we show isometric extrapolation of an infinitesimal isometric variation in Fig. 1. Beyond this, we also tested our approach on problems such as computing elastic averages, discrete geodesics (both isometric and non-isometric), and constriction or inflation of certain parts of a mesh via length constraints.

4. Reconstruction of an immersion

In the preceding sections, we discussed the geometry as well as constrained optimization problems on the NRIC manifold $\mathcal{M}$, *i.e.* in terms of edge lengths and dihedral angles. The remaining task is to reconstruct for given $z \in \mathcal{M}$ an immersion $X \in \mathcal{N}$ of the discrete surface in $\mathbb{R}^3$ with $z = Z(X)$. Additionally, one frequently asks for an approximate immersion $X \in \mathcal{N}$ for $z \notin \mathcal{M}$ such that $Z(X) \approx z$.

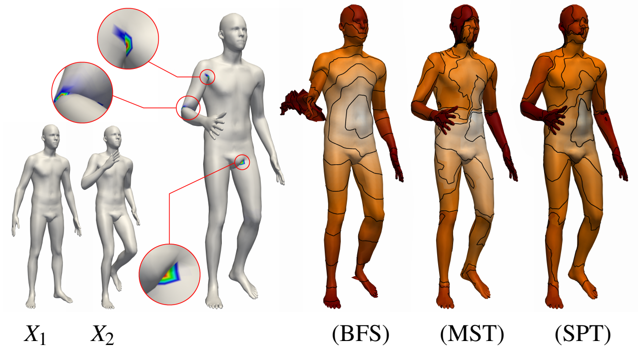

Figure 3: *Left: Input shapes X_1 and X_2 (taken from [PRMB15]) and reconstruction of linear average $(Z(X_1)+Z(X_2))/2 \notin \mathcal{M}$ with the local violations of the integrability condition as color map. Rightmost shapes: reconstruction using various spanning trees color coded with respect to the order of traversal.*

Following the proof in [WLT12], one can reconstruct the nodal positions from NRIC by traversing the dual graph and constructing triangle after triangle. To increase robustness, we traverse it along a spanning tree built such that we traverse faces with violated integrability as late as possible. To this end, each dual edge corresponds to a primal edge $e = (v, v')$ and we assign it a scalar weight reflecting the lack of integrability $w_e := \frac{|\mathrm{tr}\,\mathcal{I}_v(z) - 3| + |\mathrm{tr}\,\mathcal{I}_{v'}(z) - 3|}{2}$. We investigated using a minimal spanning tree (MST) and a shortest path tree (SPT) built from these weights and compared them against plain breadth-first search (BFS) in Fig. 3. Just applying our constructive approach works very well for $z \notin \mathcal{M}$ as long as the violations are localized as in Fig. 3. In the general case, we suggest the variational approach from [FB11] initialized with our output as post-processing.

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