

# Using Cayley-Menger Determinants for Geometric Constraint Solving

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## Abstract

*We use Cayley-Menger Determinants (CMDs) to obtain an intrinsic formulation of geometric constraints. First, we show that classical CMDs are very convenient to solve the Stewart platform problem. Second, issues like distances between points, distances between spheres, cocyclicity and cosphericity of points are also addressed. Third, we extend CMDs to deal with asymmetric problems. In 2D, the following configurations are considered: 3 points and a line; 2 points and 2 lines; 3 lines. In 3D, we consider: 4 points and a plane; 2 points and 3 planes; 4 planes.*

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling

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## 1. Introduction

The problem of solving geometric constraints often occurs in CAD, robotics, computer graphics, molecular biology, etc [Doh95, BR98]. In CAD, nowadays geometric modelers enable designers to describe geometric elements such as points, lines, circles, Bézier curves, etc in 2D and planes, quadrics, tori, Bézier patches, etc, in 3D by specifying constraints between them. Typical constraints may be: distances, angles, incidence or tangency relations. The modeler has to solve a system of constraints usually composed of polynomial equations. It decomposes the system into irreducible subsystems [HY01, GHY02], and solves them with symbolic or numerical methods such as: The Newton-Raphson iterations, homotopy-based methods, and interval analysis techniques [LM95, Dur98, HD99, Yan03]. The use of these later is less common in Geometric Constraint Solving (GCS) [JAMSR01]. However, numerical methods prevail because today symbolic packages are not powerful enough to treat 3D real world geometric problems.

In this paper, we show that using the cartesian coordinates to express equations of geometric constraints is neither the only nor the best approach of doing, we propose the use of Cayley-Menger Determinants (CMDs) instead. The rest of this section discusses some related works (subsection 1.1) and presents (subsection 1.2) the advantages of the intrinsic formulation: formulation independent from any particular coordinate system. Section 2 shows that

CMDs are much more suitable for solving the Stewart platform problem than the usual approaches which use cartesian coordinates [Dur98, HD99]. The obtained system is much simpler, without spurious roots, and easily tractable by symbolic methods. Classical CMDs [Ber90, Hav91, Blu53] are shortly presented in section 3. New CMDs, for some asymmetric problems, are proposed for 2D and 3D examples in section 4.

### 1.1. Related works

Recently, D. Lesage, P. Serré and J-C. Léon, within the framework of Serré's PhD thesis [Ser00], express all 2D constraints in a coordinate free way [LLS02]. They don't use the Cayley-Menger formalism –which proves there are several intrinsic formulations. Instead they find independent angular and vectorial loops in some constraint graphs; then each loop gives a constraint, which is translated into equations. The unknowns are not the coordinates of points, lines, vectors, etc, but norms of vectors, and angles between vectors (which, again, are not represented by their coordinates); in other words, unknowns are scalar products between vectors. This work proves that coordinate free approaches are indeed feasible, and can be realized in a systematic way. It also proves the advantages of an intrinsic approach (see section 1.2). Podgorelec in [Pod02] and Lu Yang *et al* in [ZYY94] also propose non cartesian approaches.

Finally, to prevent a very frequent confusion, note that cartesian coordinates, Grassman Plücker coordinates, pentaspheric coordinates among others are not intrinsic formulations because each of them is dependent on a particular coordinate system.

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**1.2. Advantages of the intrinsic formulation**

The intrinsic formulation has several advantages [LLS02, PTRT03]. First, under this formulation some problems become tractable with symbolic computations. Second, it naturally takes into account technical unknowns and constraints (eg. price, temperature, strength, etc). Thus it avoids the limitations of many geometric decomposition methods. Third, the qualitative study of the resulting systems of equations is straightforward: the number of equations and unknowns are equal in correct systems; the resulting system of equations can be decomposed with bipartite graph matching methods [AAJM93], and structurally irreducible subsystems can be studied with the probabilistic numerical methods [LM98]. This contrasts with the classical cartesian formulation, where the correct systems are fixed only modulo a displacement in space; such systems are called rigid; they have less equations than unknowns (coordinates): 3 in 2D (2 translations and 1 rotation), 6 in 3D (3 translations and 3 rotations); their resolution and their geometric decomposition are thus confronted by several complications. Moreover, with the non cartesian approach, the qualitative analysis can detect mistakes often hidden with the cartesian formulation.

**2. The Stewart platform problem**

Given the lengths of the 12 edges of a 3D octahedron; the Stewart platform problem, also called the octahedron problem [Dur98, HD99, NW91], is then to find compatible coordinates for the 6 vertices  $s_i, i \in [1;6]$ . The 12 edges of the octahedron are:

$$s_2s_3, s_3s_4, s_4s_5, s_5s_2, s_1s_2, s_1s_3, s_1s_4, s_1s_5, s_6s_2, s_6s_3, s_6s_4, s_6s_5$$

This problem is met in CAD as a typical irreducible 3D problem in constraint-based geometric modeling, and in robotic with the Stewart platform: the Stewart triangular platform  $s_1s_2s_3$  (Fig. 1) is driven with 6 jacks (with variable lengths)  $s_1s_4, s_1s_5, s_2s_5, s_2s_6, s_3s_6, s_3s_4$  from a ground triangular base  $s_4s_5s_6$ . Edges of the triangular platform and of the base are rigid, i.e. their lengths are constant.

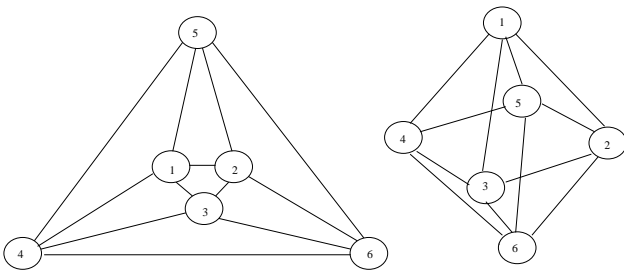


Figure 1: Two isomorphic graphs of the Stewart platform.

It is possible to use cartesian coordinates to pose the problem, but from the solving point of view choosing the more convenient coordinate system is not obvious, nowadays computational algebra packages are not powerful enough to solve the system, and a heavy work need to be done, by hand, in order to reduce the system to an irreducible system in 3 unknowns and 3 equations of degree 4 [Dur98, HD99]. The resulting system has Bezout number  $4 \times 4 \times 4 = 64$ , and BKK bound (or mixed volume) equals to 16.

Another possibility is to use CMDs. See [Ber90, Hav91] or section 3 for more details. It directly yields to 2 degree 4 equations in 2 unknowns. The CMD gives the relations between the distances between 5 points in 3D (between 4 points in 2D, between  $n + 2$  points in  $nD$ ). The CMD for the Stewart platform problem can be defined as follows:

- First, for points  $s_1$  and  $s_2, s_3, s_4, s_5$  (the equatorial square and the north vertex in the right part of Fig. 1). It gives an algebraic equation between squared distances

$$d_{12}, d_{13}, d_{14}, d_{15}, d_{23}, d_{24}, d_{25}, d_{34}, d_{35}, d_{45}$$

where  $d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$ . All these distances are known, except  $d_{24}$  and  $d_{35}$ , the squared lengths of diagonals of the equatorial square in Fig. 1. The equation has degree 4 and involves 2 unknowns.

- Second, for points  $s_6$  and  $s_2, s_3, s_4, s_5$  (the equatorial square and the south vertex in the right part of Fig. 1). It gives another 4 degree algebraic equation, with the same 2 unknowns  $d_{24}$  and  $d_{35}$ . It is obvious that this equation is generically independent of the previous one.

Thus we obtain an algebraic system in 2 unknowns and 2 equations, each of degree 4. The corresponding curves can be drawn in the plane using any standard curve plotting method. From the Bezout theorem, this system cannot have more than 16 solutions in complex projective space (i.e. taking into account multiple solutions, real and complex solutions, and solutions at infinity). Other methods yield to systems with greater Bezout number (typically 64), and in such a case it is not obvious at all to prove that there are no more than 16 solutions.

The system can be solved by any standard numerical method, say homotopy. But since there are only 2 equations in 2 unknowns, it becomes tractable with symbolic methods. For instance the Sylvester resultant, gives a degree 16 equation in only one of the unknowns. It also becomes possible to discuss degeneracies, but this question has not been investigated at this moment. Once we have the length of diagonals  $d_{24}$  and  $d_{35}$ , it is trivial to find consistent coordinates for the six vertices.

The trick here is to not to use coordinates, but to compute distances, which are independent of the coordinate system (once the scale, say meters or millimeters, has been chosen). Other parameters independent of coordinates system are angles and cross ratios, and they may be more convenient in other cases.

**3. Classical Cayley-Menger determinants**

This section presents an introduction to classical CMDs. See [Ber90, Hav91] for more details.

**3.1. Distances between points**

Given 5 points in the Euclidean 3D space, the following relation holds, between all their squared distances:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12} & d_{13} & d_{14} & d_{15} \\ 1 & d_{21} & 0 & d_{23} & d_{24} & d_{25} \\ 1 & d_{31} & d_{32} & 0 & d_{34} & d_{35} \\ 1 & d_{41} & d_{42} & d_{43} & 0 & d_{45} \\ 1 & d_{51} & d_{52} & d_{53} & d_{54} & 0 \end{vmatrix} = 0$$

where  $d_{ij} = (p_i - p_j) \cdot (p_i - p_j)$  is the square of the distance between points  $i$  and  $j$ .  $|M|$  is the so called CMD.

In order to save spaces in equation writings, we define, for the rest of the paper, values  $v_i = x_i^2 + y_i^2 + z_i^2$  with  $i \in [1;6]$ .

*Proof*  $M = AB^t$  where  $A$  and  $B^t$  are matrices with rank at most equal to 5 ( $A$  and  $B$  have 6 rows but only 5 columns), so:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & 2x_1 & 2y_1 & 2z_1 & 1 \\ v_2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ v_3 & 2x_3 & 2y_3 & 2z_3 & 1 \\ v_4 & 2x_4 & 2y_4 & 2z_4 & 1 \\ v_5 & 2x_5 & 2y_5 & 2z_5 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & -x_1 & -y_1 & -z_1 & v_1 \\ 1 & -x_2 & -y_2 & -z_2 & v_2 \\ 1 & -x_3 & -y_3 & -z_3 & v_3 \\ 1 & -x_4 & -y_4 & -z_4 & v_4 \\ 1 & -x_5 & -y_5 & -z_5 & v_5 \end{pmatrix}$$

Actually,  $|M|$  still vanishes when points in  $A$  and points in  $B$  are not the same. It gives another non trivial relation for distances between two point sets  $P_i$  and  $Q_j$  for  $i, j \in [1;5]$  (the diagonal entries in  $M$  are no more zeros. They represent squared distances between  $P_i$  and  $Q_i$ ).  $\square$

The previous determinant can be extended to  $2D, 4D$ , etc. Finally let us mention that the CMD is equal to a signed volume, up to some multiplicative constant.

### 3.2. Distances between spheres

In  $3D$ , one can define the signed distance (or power) of 2 spheres  $S_i = (x_i \ y_i \ z_i)$  with radius  $R_i$  and  $S_j = (x_j \ y_j \ z_j)$  with radius  $R_j$  as:

$$K_{ij} = K_{ji} = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - (R_i^2 + R_j^2)$$

Actually, this signed distance does not depend on the system of coordinates used (once the scale is chosen, say meter or millimeter). Then the distances between any six spheres in  $3D$  fulfill:

$$|K| = \begin{vmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{vmatrix} = 0$$

*Proof*  $K = AB^t$  and  $A$  and  $B$  have rank at most equal to 5 (6 rows, 5 columns), so:

$$A = \begin{pmatrix} v_1 - R_1^2 & x_1 & y_1 & z_1 & 1 \\ v_2 - R_2^2 & x_2 & y_2 & z_2 & 1 \\ v_3 - R_3^2 & x_3 & y_3 & z_3 & 1 \\ v_4 - R_4^2 & x_4 & y_4 & z_4 & 1 \\ v_5 - R_5^2 & x_5 & y_5 & z_5 & 1 \\ v_6 - R_6^2 & x_6 & y_6 & z_6 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -2x_1 & -2y_1 & -2z_1 & v_1 - R_1^2 \\ 1 & -2x_2 & -2y_2 & -2z_2 & v_2 - R_2^2 \\ 1 & -2x_3 & -2y_3 & -2z_3 & v_3 - R_3^2 \\ 1 & -2x_4 & -2y_4 & -2z_4 & v_4 - R_4^2 \\ 1 & -2x_5 & -2y_5 & -2z_5 & v_5 - R_5^2 \\ 1 & -2x_6 & -2y_6 & -2z_6 & v_6 - R_6^2 \end{pmatrix}$$

$\square$

This relation also holds when some radii are 0. It is then possible to compute the relation between any point and any 5 spheres in  $\mathbb{R}^3$ .

### 3.3. Cocyclicity or cosphericity of points

It is possible to express the cocyclicity of 4 points in  $2D$ , or the cosphericity of 5 points in  $3D$ , of  $d + 2$  points in  $\mathbb{R}^d$  without coordinates, just by using squared distances between points.

In  $2D$ , 4 points are cocyclic (belong to the same circle) iff:

$$|C| = \begin{vmatrix} 0 & d_{12} & d_{13} & d_{14} \\ d_{21} & 0 & d_{23} & d_{24} \\ d_{31} & d_{32} & 0 & d_{34} \\ d_{41} & d_{42} & d_{43} & 0 \end{vmatrix} = 0$$

where  $d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2$  is the squared distance between points  $i$  and  $j$ , and thus is independent of cartesian systems. This is equivalent to the Ptolemy theorem [Coo71].

*Proof* let  $(x_0, y_0)$  be the center and  $R_0$  be the radius of the (unknown) circle. We have, in some cartesian frame (we will remove this dependency later):  $(x_i - x_0)^2 + (y_i - y_0)^2 - R_0^2 = 0$  for  $i = 1, 2, 3, 4$ . We can express these conditions this way:

$$\begin{pmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2x_0 \\ -2y_0 \\ x_0^2 + y_0^2 - R_0^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

It can be seen as a linear homogeneous system with unknowns in the column vector. There is a non zero solution iff the determinant of the matrix (call it  $C_1$ ) is zero. We have a condition for cocyclicity, but it depends on the cartesian frame.

We can also express the system this way:

$$\begin{pmatrix} 1 & -2x_1 & -2y_1 & x_1^2 + y_1^2 \\ 1 & -2x_2 & -2y_2 & x_2^2 + y_2^2 \\ 1 & -2x_3 & -2y_3 & x_3^2 + y_3^2 \\ 1 & -2x_4 & -2y_4 & x_4^2 + y_4^2 \end{pmatrix} \begin{pmatrix} x_0^2 + y_0^2 - R_0^2 \\ x_0 \\ y_0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Here again, the determinant of the matrix (call it  $C_2$ ) must vanish. Now remark that  $C = C_1 C_2^t$ . Thus the determinant of  $C = C_1 C_2^t$  must also vanish. We have proved the cocyclicity condition. This relation can be easily extended to  $\mathbb{R}^3$  and beyond.  $\square$

**4. New Cayley-Menger determinants**

Classical CMDs apply to very symmetric problems. Nevertheless typical problems, in CAD and constraint-based geometric modeling, are not as symmetrical as the Stewart platform problem. Constraints involve heterogeneous data: points, planes, lines, spheres ... Here are simple examples of heterogeneous CMDs.

**4.1. 3 points and 1 line in 2D**

In 2D, consider 3 points  $s_1, s_2, s_3$  and a line  $l$ . Let  $d_{ij}$  for  $i, j = 1, 2, 3$  be the squared distances between points  $s_i$  and  $s_j$ , and let  $d_i$  be the signed (non squared) distance between point  $s_i$  and line  $l: d_i = ax_i + by_i + c$ . Assuming  $l$  has equation:  $ax + by + c = 0$  with  $a^2 + b^2 = 1$ , in the cartesian frame we want to get rid of.

Due to coplanarity, there is a relation between the  $d_{ij}$  and the  $d_i$  (in passing, there is only one equality: the configuration involves 6 distances but has only five "degrees of freedom". Other possible constraints, like triangular inequalities for the triangle to be realizable; are not considered). This relation may seem a bit strange:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & d_{12} & d_{13} & d_1 \\ 1 & d_{21} & 0 & d_{23} & d_2 \\ 1 & d_{31} & d_{32} & 0 & d_3 \\ 0 & d_1 & d_2 & d_3 & \frac{-1}{2} \end{vmatrix} = 0$$

where diagonal zeros stand for  $d_{ii}$  and  $d_{ij} = d_{ji}$  of course. Note that  $M$  is symmetric despite the dissymmetry of the problem.

*Proof* Just check below that  $M$  is the product of the following  $5 \times 4$  and  $4 \times 5$  matrices (thus with rank 4, generically):

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1^2 + y_1^2 & 2x_1 & 2y_1 & 1 \\ x_2^2 + y_2^2 & 2x_2 & 2y_2 & 1 \\ x_3^2 + y_3^2 & 2x_3 & 2y_3 & 1 \\ c & -a & -b & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & -x_1 & -x_2 & -x_3 & \frac{a}{2} \\ 0 & -y_1 & -y_2 & -y_3 & \frac{b}{2} \\ 1 & x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 & c \end{pmatrix}$$

□

**4.2. 2 points and 2 lines in 2D**

In 2D, consider 2 points  $s_1$  and  $s_2$  and 2 lines  $l_1$  and  $l_2$ . Call  $s_i l_j$  with  $i, j = 1, 2$  the distance between point  $s_i$  and line  $l_j$ . Lines  $l_j$  have equations  $a_j x + b_j y + c_j = 0$ , in the cartesian frame we want to eliminate, we suppose for simplicity that  $a_j^2 + b_j^2 = 1$ . Thus  $s_i l_j = a_j x_i + b_j y_i + c_j$ . Call  $l_1 l_2$  the "distance", actually the cosine, between the 2 line directions:  $l_1 l_2 = a_1 a_2 + b_1 b_2$ . Call  $s_1 s_2$  the squared distance between  $s_1$  and  $s_2$ . The relation between these distances is given by the nullity of the non symmetric determinant:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & s_1 s_2 & 2s_1 l_1 & 2s_1 l_2 \\ 1 & s_1 s_2 & 0 & 2s_2 l_1 & 2s_2 l_2 \\ 0 & -s_1 l_1 & -s_2 l_1 & 1 & l_1 l_2 \\ 0 & -s_1 l_2 & -s_2 l_2 & l_1 l_2 & 1 \end{vmatrix} = 0$$

*Proof* The  $5 \times 5$  matrix  $M$  is the product of the following  $5 \times 4$  and  $4 \times 5$  matrices, which have ranks at most equal to 4, so the rank of matrix  $M$  is also at most equal to 4:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1^2 + y_1^2 & 2x_1 & 2y_1 & 1 \\ x_2^2 + y_2^2 & 2x_2 & 2y_2 & 1 \\ -c_1 & a_1 & b_1 & 0 \\ -c_2 & a_2 & b_2 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & -x_1 & -x_2 & a_1 & a_2 \\ 0 & -y_1 & -y_2 & b_1 & b_2 \\ 1 & x_1^2 + y_1^2 & x_2^2 + y_2^2 & 2c_1 & 2c_2 \end{pmatrix}$$

Note that  $|M|$  is not identically zero (i.e. we can find entries such that  $|M|$  does not vanish), since we can find in it a perfect matching (i.e. one generically non zero element in each and every row and column). □

**4.3. 3 lines in 2D**

Let  $l_i, i = 1, 2, 3$  be any 3 lines in 2D having equations:  $a_i x + b_i y + c_i = 0$ . Assume without lose of generality that  $a_i^2 + b_i^2 = 1$ . Let  $c_{ij} = c_{ji} = a_i a_j + b_i b_j$  be the cosine of the angle between  $l_i$  and  $l_j$ . As well known, they fulfill:

$$\begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{21} & 1 & c_{23} \\ c_{31} & c_{32} & 1 \end{vmatrix} = 0$$

since  $\begin{pmatrix} 1 & c_{12} & c_{13} \\ c_{21} & 1 & c_{23} \\ c_{31} & c_{32} & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$

the ranks of these 2 last matrices is equal to 2. For  $d + 1$  vectors to belong to the same vectorial space of dimension  $d$ , their Gram matrix (the matrix of their scalar product [Ber90]) must have rank  $d$ , thus the determinant must vanish.

**4.4. 4 points and 1 plane in 3D**

In 3D, consider 4 points  $s_i, i \in [1; 4]$  and 1 plane  $p$  with equation:  $ax + by + cz + d = 0$ , where  $a^2 + b^2 + c^2 = 1$ . The squared distance between two points  $s_i$  and  $s_j$  is  $d_{ij}$  and the signed distance between  $s_i$  and  $p$  is  $d_i = ax_i + by_i + cz_i + d$ .

This is the relation between all these distances:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & d_{12} & d_{13} & d_{14} & d_1 \\ 1 & d_{21} & 0 & d_{23} & d_{24} & d_2 \\ 1 & d_{31} & d_{32} & 0 & d_{34} & d_3 \\ 1 & d_{41} & d_{42} & d_{43} & 0 & d_4 \\ 0 & d_1 & d_2 & d_3 & d_4 & \frac{-1}{2} \end{vmatrix} = 0$$

Proof

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & 2x_1 & 2y_1 & 2z_1 & 1 \\ v_2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ v_3 & 2x_3 & 2y_3 & 2z_3 & 1 \\ v_4 & 2x_4 & 2y_4 & 2z_4 & 1 \\ d & -a & -b & -c & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -x_1 & -x_2 & -x_3 & -x_4 & a/2 \\ 0 & -y_1 & -y_2 & -y_3 & -y_4 & b/2 \\ 0 & -z_1 & -z_2 & -z_3 & -z_4 & c/2 \\ 1 & v_1 & v_2 & v_3 & v_4 & d \end{pmatrix}$$

□

### 4.5. 3 points and 2 planes in 3D

Consider 3 points  $s_1, s_2, s_3$  and 2 planes  $p_1$  and  $p_2$  in 3D. Assume that  $p_i$  has equation:  $a_i x + b_i y + c_i z + d_i = 0$  in some coordinate frame we want to get rid of, with  $a_i^2 + b_i^2 + c_i^2 = 1$ . Note  $s_i p_j$  the signed distance between point  $s_i$  and plane  $p_j$ :  $s_i p_j = a_j x_i + b_j y_i + c_j z_i + d_j$ , and note  $p_i p_j$  the cosine of the angle between  $p_i$  and  $p_j$ :  $p_i p_j = a_i a_j + b_i b_j + c_i c_j$ .

This is the relation between all these distances:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & s_1 s_2 & s_1 s_3 & 2s_1 p_1 & 2s_1 p_2 \\ 1 & s_1 s_2 & 0 & s_2 s_3 & 2s_2 p_1 & 2s_2 p_2 \\ 1 & s_1 s_3 & s_2 s_3 & 0 & 2s_3 p_1 & 2s_3 p_2 \\ 0 & -s_1 p_1 & -s_2 p_1 & -s_3 p_1 & 1 & p_1 p_2 \\ 0 & -s_1 p_2 & -s_2 p_2 & -s_3 p_2 & p_1 p_2 & 1 \end{vmatrix} = 0$$

Proof

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & 2x_1 & 2y_1 & 2z_1 & 1 \\ v_2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ v_3 & 2x_3 & 2y_3 & 2z_3 & 1 \\ -d_1 & a_1 & b_1 & c_1 & 0 \\ -d_2 & a_2 & b_2 & c_2 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -x_1 & -x_2 & -x_3 & a_1 & a_2 \\ 0 & -y_1 & -y_2 & -y_3 & b_1 & b_2 \\ 0 & -z_1 & -z_2 & -z_3 & c_1 & c_2 \\ 1 & v_1 & v_2 & v_3 & 2d_1 & 2d_2 \end{pmatrix}$$

□

### 4.6. 2 points and 3 planes in 3D

In the same way as above, distances between 2 points  $s_1, s_2$  and 3 planes  $p_1, p_2, p_3$  in 3D are linked by the following relation:

$$|M| = \begin{vmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & s_1 s_2 & 2s_1 p_1 & 2s_1 p_2 & 2s_1 p_3 \\ 1 & s_1 s_2 & 0 & 2s_2 p_1 & 2s_2 p_2 & 2s_2 p_3 \\ 0 & -s_1 p_1 & -s_2 p_1 & 1 & p_1 p_2 & p_1 p_3 \\ 0 & -s_1 p_2 & -s_2 p_2 & p_1 p_2 & 1 & p_2 p_3 \\ 0 & -s_1 p_3 & -s_2 p_3 & p_1 p_3 & p_2 p_3 & 1 \end{vmatrix} = 0$$

Proof

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & 2x_1 & 2y_1 & 2z_1 & 1 \\ v_2 & 2x_2 & 2y_2 & 2z_2 & 1 \\ -d_1 & a_1 & b_1 & c_1 & 0 \\ -d_2 & a_2 & b_2 & c_2 & 0 \\ -d_3 & a_3 & b_3 & c_3 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -x_1 & -x_2 & a_1 & a_2 & a_3 \\ 0 & -y_1 & -y_2 & b_1 & b_2 & b_3 \\ 0 & -z_1 & -z_2 & c_1 & c_2 & c_3 \\ 1 & v_1 & v_2 & 2d_1 & 2d_2 & 2d_3 \end{pmatrix}$$

□

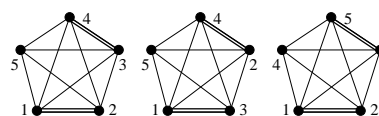


Figure 2: Isomorphic subgraphs of the same class monomials

### 4.7. 4 planes in 3D

Like in section 4.3, we only need to make the determinant of the Gram matrix of 4 plane normals in 3D vanish.

## 5. Future extensions

A first open problem is to find relations involving also lines in 3D, and not only points and planes. May be Grassman Plücker coordinates for lines in some cartesian frame must be used, before the frame elimination. One such relation, due to Neil White, is given in Sturmfels's book [Stu93], th. 3.4.7: it is the condition for five lines in 3D space to have a common transversal line. Philippe Serré, in his PhD thesis [Ser00], also gives the relation involving distances between two lines  $AB$  and  $CD$  and between points  $A, B, C, D$ .

A second problem is to find such polynomial relations. From a theoretical point of view, it suffices to use a Grobner package to eliminate variables representing coordinates in some set of equations (for instance equations:  $(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - d_{ij}^2 = 0, i \in [1;4], j \in [i+1;5]$ , to find the Cayley-Menger equation relating distances between 5 points in 3D). In practice, Grobner packages are not powerful enough. The polynomial condition can be computed by interpolation: for instance, to guess the Cayley-Menger equation in 3D, one can proceed in three steps:

- Generate  $N$  random configurations of 5 points  $(x_i, y_i, z_i) \in \mathbb{Z}^3$ ,
- Compute square distances  $d_{ij}^k, i \in [1;4]$  and  $j \in [i+1;5]$  for each configuration  $k \in [1;N]$ . This gives  $N$  15D points.
- All these  $N$  15D points lie on the zero-set of an unknown polynomial in the variables  $d_{ij}$ . We search for this polynomial by trying increasing degrees.

This polynomial has an exponential number of monomials, so there is an exponential number of unknown coefficients. However, due to the symmetry some monomials have the same coefficients and are said to lie in the same "class". For instance monomials  $d_{12}^2 d_{34}^2$ ,

$d_{13}^2 d_{24}^2$ , etc lie in the same class: Monomials of the same class correspond to isomorphic edge weighted subgraphs of  $K_5$ , the complete graph with 5 vertices and with edges weighted by the degree of the corresponding monomial (Fig. 2). To be feasible this approach must exploit this symmetry to reduce the number of unknown coefficients to the number of classes. The fast generation of these classes (and of one instance per class) is an interesting and non trivial combinatorial problem by itself, related to the Polya's counting theory.

To validate this approach we implemented a simple algorithm that computes Cayley-Menger relations and distance relations for 6 2D points to lie on the same conic as well as for 10 3D points to lie on the same quadric. We noticed that this first implementation works slowly because it doesn't exploit the symmetry. Moreover its output (the polynomial coefficients) has an exponential size and is thus unusable. Exploiting symmetry is thus essential.

## 6. Conclusion

This paper has shown that CMDs may give simpler algebraic systems, with less spurious roots, and tractable with today's symbolic algebra packages. Examples of points/circles, circles/circles and spheres/spheres relations are given. Unfortunately, these classical CMDs involve only relations between geometric primitives of the same type. This paper also introduced new CMDs formulations to find relations between heterogeneous 2D and 3D geometric entities. It remains to propose an automatic method to generate such new Cayley-Menger relations: a new challenging problem for computational algebra.

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## References

- [AAJM93] AIT-AOUDIA S., JEGOU R., MICHELUCCI D.: Reduction of constraint systems. In *Compugraphic* (Alvor, Portugal, 1993), pp. 83–92. 2
- [Ber90] BERGER M.: *Géométrie*. Nathan, 1990. 1, 2, 4
- [Blu53] BLUMENTHAL L.: *Theory and Applications of Distance geometry*. Clarendon Press, Oxford, 1953. 1
- [BR98] BRUDERLIN B., ROLLER D. (Eds.): *Geometric Constraint Solving and Applications*. Springer, 1998. 1
- [Coo71] COOLIDGE J. L.: *A Treatise on the Geometry of the Circle and Sphere*. Chelsia, New York, 1971. 3
- [Doh95] DOHMEN M.: A survey of constraint satisfaction techniques for geometric modeling. *Computers and Graphics* 19, 6 (1995), 831–845. 1
- [Dur98] DURAND C. B.: *Symbolic and Numerical Techniques for Constraint Solving*. PhD thesis, Purdue University, 1998. 1, 2
- [GHY02] GAO X.-S., HOFFMANN C. M., YANG W.-Q.: Solving spatial basic geometric constraint configurations with locus intersection. In *ACM Symposium on Solid Modeling and Applications* (Saarbrücken, Germany, 2002), ACM Press, pp. 95–104. 1
- [Hav91] HAVEL T. F.: Some examples of the use of distances as coordinates for euclidean geometry. *Journal of Symbolic Computation*, 11 (1991), 579–593. 1, 2
- [HD99] HOFFMANN C., DURAND C.: Variational constraints in 3d. In *Intl. Conf. on Shape Modeling and Applications* (Aizu, Japan, 1999), pp. 90–97. 1, 2
- [HY01] HOFFMANN C. M., YUAN B.: On spatial constraint solving approaches. In *Proc. of Automated Deduction in Geometry 2000, ETH Zurich* (2001), Richter-Gebert J., Wang D., (Eds.), pp. 1–15. 1
- [JAMSR01] JOAN-ARINYO R., MATA N., SOTO-RIERA A.: A constraint solving-based approach to analyze 2d geometric problems with interval parameters. In *Proc. 6th ACM symposium on Solid modeling and applications* (2001), ACM Press, pp. 11–17. 1
- [LLS02] LESAGE D., LÉON J.-C., SERRÉ P.: A declarative approach to a 2d variational modeler. In *ID-MME'2000* (2002), Kluwer, pp. 105–112. 1, 2
- [LM95] LAMURE H., MICHELUCCI D.: Solving constraints by homotopy. In *Symp. on Solid Modeling Foundations and CAD/CAM Applications* (May 1995), pp. 263–269. 1
- [LM98] LAMURE H., MICHELUCCI D.: Qualitative study of geometric constraints. In *Geometric Constraint Solving and Applications* (1998), Bruderlin B., Roller D., (Eds.), Springer Verlag, pp. 234–258. 2
- [NW91] NANUA P., WALDRON K.: Direct kinematic solution of a stewart platform. *IEEE Trans. on Robotics and Automation* 6, 4 (1991), 438–444. 2
- [Pod02] PODGORELEC D.: A new constructive approach to constraint-based geometric design. *Computer-Aided Design* 34, 11 (September 2002), 769–785. 1
- [PTRT03] PORTA J. M., THOMAS F., ROS L., TORRAS C.: A branch-and-prune algorithm for solving systems of distance constraints. In *IEEE Int'l Conf. on Robotics and Automation* (Taipei, Taiwan, 2003), pp. 342–347. 2
- [Ser00] SERRÉ P.: *Cohérence de la spécification d'un objet de l'espace euclidien à n dimensions*. PhD thesis, Ecole Centrale Paris, 2000. 1, 5
- [Stu93] STURMFELS B.: *Algorithms in Invariant Theory*. Springer, 1993. 5
- [Yan03] YANG L.: Solving geometric constraints with distance-based global coordinate system. In *Int'l Workshop on Geometric Constraint Solving* (Beijing, China, 2003). 1
- [ZYY94] ZHANG J. Z., YANG L., YANG X. C.: The realization of elementary configurations in euclidean space. *Science in China A* 37, 1 (1994), 15–26. 1